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# Continuous Truth II: Reflections

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**Abstract.** In the late 1960s, Dana Scott first showed how the Stone-Tarski topological interpretation of Heyting’s calculus could be extended to model intuitionistic analysis; in particular Brouwer’s continuity principle. In the early ’80s we and others outlined a general treatment of non-constructive objects, using *sheaf models*—constructions from topos theory—to model not only Brouwer’s non-classical conclusions, but also his creation of “new mathematical entities”. These categorical models are intimately related to, but more general than Scott’s topological model.

The primary goal of this paper is to consider the question of iterated extensions. Can we derive new insights by repeating the second act?

In *Continuous Truth I*, presented at Logic Colloquium ’82 in Florence, we showed that general principles of continuity, local choice and local compactness hold in the gros topos of sheaves over the category of separable locales equipped with the open cover topology.

We touched on the question of iteration. Here we develop a more general analysis of iterated categorical extensions, that leads to a reflection schema for statements of predicative analysis.

We also take the opportunity to revisit some aspects of both Continuous Truth I and Formal Spaces (Fourman & Grayson 1982), and correct two long-standing errors therein.

**Keywords:** sheaf model, logic, intuitionism, predicative, analysis, topos.

## 1 Introduction

Brouwer, in his Cambridge lectures [8], distinguishes two “acts of intuitionism”. The first (p. 4) is to reject some “principles of classical logic, blindly formulated.” In particular, Brouwer rejects the *principium tertii exclusi*: “the principle of the excluded third, ... cannot in general serve as a principle for discovering mathematical truths.” This first act is formally enshrined in Heyting’s predicate calculus, which intuitionism shares with various flavours of constructive mathematics.

Brouwer’s SECOND ACT OF INTUITIONISM is more subtle.

*Admitting two ways of creating new mathematical entities: firstly in the shape of more or less freely proceeding infinite sequences of mathematical entities previously acquired; secondly in the shape of mathematical species, ...*  
(*op cit.* p.8)

Brouwer uses such non-constructive creations to derive strongly non-classical results, such as his celebrated continuity principle:

*Each full function of the unity continuum is uniformly continuous.* (p.80)

To model this ‘second act’, we base ourselves in a constructive setting  $\mathbb{B}$ , and model the addition of new mathematical entities by the passage to an extension  $\mathbb{E} = \mathbb{B}[D]$ , that includes a new entity,  $D$ . Working within the extension, we model Brouwer’s arguments, and his non-classical conclusions—such as the continuity principle.

The Lawvere-Tierney notion of an *elementary topos*  $\mathbb{E}$  provides a paradigmatic example of a constructive setting, and their construction of a classifying topos, extending a base topos by adding a generic model,  $D$ , of some geometric theory, has now been widely used to model the introduction of new mathematical entities (see e.g. [9] for a recent example).

For the simplest infinitary extensions — adding a generic infinite sequence by taking sheaves over formal Baire space or formal Cantor space [1, 2] — it is easy to see that the construction is reflexive. This was part of the folklore thirty years ago, but appears to be still unrecorded in the literature. We first review these examples, and then consider models such as those introduced in [3–6] and used extensively by, e.g., [7, 9].

In [6], we considered the interpretation of logic in the gros topos of sheaves over the category of separable locales equipped with the open cover topology. We showed that general principles of continuity, local choice and local compactness hold for these models. In §5 we touched on the question of iteration. Our analysis there focussed on low-level detail. We failed to see the wood for the trees. Plans, announced there, to develop a high-level account in collaboration with Max Kelly never materialised.

Here we provide a quite general category-theoretic account of iteration — the construction of a model within the model — a preliminary report of ongoing work. This allows us to show that in some models,  $\mathfrak{M}$ , a reflection principle that states that, *a statement  $\phi$  of predicative analysis is true iff it is true in the model*, is valid:

$$\text{Reflection for } \phi : \quad \mathfrak{M} \models \phi \quad \text{iff} \quad \mathfrak{M} \models \ulcorner \mathfrak{M} \models \phi \urcorner$$

We also present the basic facts we need relating open locales to open maps, correcting an error in [10]. These facts are no longer new — for an elephantine account see [11] (🐘) — but our presentation maybe more accessible, from a logical perspective, than the definitive treatment in [12].

## 2 Preliminaries

We compose morphisms in diagram order: for  $a \xrightarrow{f} b \xrightarrow{g} c$  we have  $a \xrightarrow{fg} c$ . Otherwise, our notations and definitions generally follow those of Mac Lane & Moerdijk [13] (M&M) or Johnstone (🐼), except where some constructive finesse is required.

*Context.* Our arguments are intended always to be formalisable in Higher-order Heyting Arithmetic (HAH), a simple impredicative type theory also known as the logic of topoi. Much, maybe all, of what we say will be formalisable within a weaker predicative setting [9, 14], but we have neither space nor time to attempt that here. A set (Kuratowski) *finite* iff it can be enumerated by some natural number, and *countable* iff it can be enumerated by  $\mathbb{N}$ ; in each case, repetitions are allowed. Any countable  $X$  is *inhabited* — which means that there is some  $x$  such that  $x \in X$ .

We use the locutions of dependent types, for example when we discuss coverings and sheaves, but these can be interpreted within a simple type theory using a standard categorical trick, due originally to Grothendieck. An indexed type  $A_i \mid i \in I$  is given by a morphism  $A \longrightarrow I$ . This representation means that operations are defined uniformly across the family. Jean Bénabou and his school showed how it can be used to develop category theory in an essentially predicative setting [15].

## 2.1 Frames and Locales

We recap some facts, which should be well-known [10, 12, 14, 16–19].

**Definition 1.** A frame,  $\mathcal{F}$ , is a complete  $\wedge \vee$ -distributive lattice; finite meets distribute over arbitrary joins:  $a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)$ . A frame morphism preserves  $\top \wedge \bigvee$ . A basis  $\mathcal{G} \subseteq \mathcal{F}$  is a subset such that  $u = \bigvee \{v \in \mathcal{G} \mid v \leq u\}$ , for every  $u \in \mathcal{F}$ . The category  $\mathcal{L}$  of locales is defined to be the opposite of the category of frames. Following [12] we often refer to its objects as spaces.

Given  $f : \mathcal{X} \longrightarrow \mathcal{Y}$ , a morphism in  $\mathcal{L}$ , the corresponding frame morphism is the inverse image morphism  $f^* : \mathcal{O}(\mathcal{Y}) \longrightarrow \mathcal{O}(\mathcal{X})$ .  $\mathcal{L}$  can be viewed as a category of generalised spaces [19]. Any set  $X$ , can be viewed as a discrete space corresponding to the frame  $\mathcal{P}(X)$ . The one-point space  $\mathbb{1} = \{*\}$  corresponds to the frame  $\mathcal{P}(\mathbb{1})$ . A *point* of  $\mathcal{X}$  is a morphism  $x : \mathbb{1} \longrightarrow \mathcal{X}$ . Classically,  $\mathcal{P}(\mathbb{1})$  appears trivial; constructively it encapsulates the ambient propositional logic,

For  $\mathcal{U} \subseteq \mathbb{1}$  we have  $\mathcal{U} = \bigvee \{\top \mid * \in \mathcal{U}\}$  ( $\{\mathbb{1}\}$  is a basis). Since this join must be preserved, for any locale,  $\mathcal{X}$ , there is a unique frame morphism,

$$\hat{\cdot} : \mathcal{P}(\mathbb{1}) \longrightarrow \mathcal{O}(\mathcal{X}), \text{ given by } \hat{\mathcal{U}} = \bigvee \{\top \mid * \in \mathcal{U}\}, \quad (1)$$

and thus a unique locale morphism  $\mathcal{X} \longrightarrow \mathbb{1}$  to the one-point space.

**Open Maps.** Any frame provides a model of Heyting’s propositional calculus. Heyting’s implication is given by,  $p \Rightarrow q = \bigvee \{r \mid r \wedge p \leq q\}$ . Heyting’s implication  $\Rightarrow$  is not, in general preserved by a frame morphism. Frame morphisms that do preserve  $\Rightarrow$  correspond to open maps of locales. They also have a simple logical characterisation [12].

**Definition 2.** A locale morphism  $f : \mathcal{A} \rightarrow \mathcal{B}$  is defined to be: surjective iff  $f^*$  is 1-1; injective iff  $f^*$  is onto; open [12, 20]<sup>1</sup> iff  $f^*$  preserves both  $\bigwedge$  and  $\Rightarrow$ .

A locale  $\mathcal{A}$  is said to be open (and surjective) iff the locale morphism  $\mathcal{A} \rightarrow *$  is open (and surjective).

*Remark 1.* ([12] V.1; ☹️ A Lemma 1.5.8) Since a frame morphism  $f^*$  preserves  $\bigvee$ , it has a right adjoint,  $f_*$ , given by  $f_*A = \bigvee\{B \mid f^*B \leq A\}$ . Dually,  $f^*$  preserves  $\bigwedge$ , iff it has a left adjoint,  $f_!$ , given by  $f_!A = \bigwedge\{B \mid A \leq f^*B\}$ , in which case  $f^*$  preserves  $\Rightarrow$  iff  $f_!$  satisfies the *Frobenius Condition*:  $f_!(A \wedge f^*(B)) = f_!A \wedge B$ .

**Lemma 1.** The locale morphism  $\mathcal{A} \rightarrow *$  is open iff it preserves  $\bigwedge$ .

*Proof.* (c.f. [12] Chapter V 3.1) For  $\mathcal{U}, \mathcal{V} \subseteq \mathbb{1}$

$$\mathcal{U} \leq \mathcal{V} \quad \text{iff} \quad * \in \mathcal{U} \rightarrow * \in \mathcal{V} \quad \text{iff} \quad \mathcal{U} = \mathbb{1} \rightarrow \mathcal{V} = \mathbb{1} \quad (2)$$

Assuming we have a left adjoint  $! \dashv \hat{\cdot}$ , so that  $!U \leq p$  iff  $U \leq \hat{p}$ , obviously  $\widehat{p \Rightarrow q} \leq \hat{p} \Rightarrow \hat{q}$ . It remains to show  $\hat{p} \Rightarrow \hat{q} \leq \widehat{p \Rightarrow q}$ . Equivalently it suffices to show, assuming  $U \wedge \hat{p} \leq \hat{q}$  that  $U \leq \widehat{p \Rightarrow q}$ . Now the following are equivalent:

$$U \leq \widehat{p \Rightarrow q} \quad \text{iff} \quad !U \leq p \Rightarrow q \quad \text{iff} \quad p \wedge !U \leq q.$$

Assuming  $U \wedge \hat{p} \leq \hat{q}$ , we apply (2) to show the last of these. If  $p \wedge !U = \mathbb{1}$  then  $p = \mathbb{1}$ , so  $\hat{p} = \top$ . Substituting  $\top$  for  $\hat{p}$  in our assumption tells us that  $U \leq \hat{q}$ , so  $!U \leq q$ ; but we also know that  $!U = \mathbb{1}$ , so  $q = \mathbb{1}$ .  $\square$

As an exercise in this form of constructive argument, we give a direct proof of the Frobenius condition.

**Lemma 2.** (☹️ p. 618) If the inverse image of locale map  $\mathcal{A} \rightarrow *$  has a left adjoint,  $! \dashv \hat{\cdot}$  then it satisfies the Frobenius condition  $!(U \wedge \hat{p}) = !U \wedge p$ .

*Proof.* By adjointness,  $!(U \wedge \hat{p}) \leq !U \wedge p$ . To show equality, assume  $!U \wedge p = \mathbb{1}$  then  $!U = \mathbb{1}$  and  $p = \mathbb{1}$  so  $\hat{p} = \top$  and  $!(U \wedge \hat{p}) = !U = \mathbb{1}$ .  $\square$

Any frame,  $\mathcal{O}(\mathcal{X})$ , can be used to provide an  $\mathcal{O}(\mathcal{X})$ -valued interpretation, as in [16], of the impredicative higher-order logic (HAH). This is the interpretation of HAH in the localic topos,  $\text{Sh}(\mathcal{X})$ , of sheaves on  $\mathcal{X}$ .

*Example 1.* Given a locale morphism  $\pi_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{X}$  we define an  $\mathcal{O}(\mathcal{X})$ -valued poset  $\mathcal{O}(\mathcal{A}/_{\mathcal{X}})$  with underlying set  $\mathcal{O}(\mathcal{A})$ . For  $U, V \in \mathcal{O}(\mathcal{A})$  we define

$$\llbracket U = V \rrbracket = \bigvee \{p \in \mathcal{O}(\mathcal{X}) \mid U \wedge \pi_{\mathcal{A}}^*(p) = V \wedge \pi_{\mathcal{A}}^*(p)\} \quad (3)$$

$$\llbracket U \leq V \rrbracket = \bigvee \{p \in \mathcal{O}(\mathcal{X}) \mid U \wedge \pi_{\mathcal{A}}^*(p) \leq V \wedge \pi_{\mathcal{A}}^*(p)\} \quad (4)$$

<sup>1</sup> Our earlier paper on Formal Spaces [10] betrayed an unfortunate confusion: our Definition 2.9 of *open map* omitted the Frobenius condition. We are grateful to the eagle-eyed Peter Johnstone for pointing this out in his review, MR0717242 (85c:03023). The statement of Theorem 2 (below) appears already as Lemma 2.12 of [10], but in the context of this weaker definition of ‘open’—thus making a weaker claim. Lemma 1 provides the necessary buttress to our earlier proof.

This  $\mathcal{O}(\mathcal{X})$ -valued poset can be viewed a frame within the  $\mathcal{O}(\mathcal{X})$ -valued interpretation. In fact, every internal frame in a localic topos arises in this way [10]. Given  $\pi_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{X}$ , a map  $f^* : \mathcal{O}(\mathcal{A}) \rightarrow \mathcal{O}(\mathcal{B})$  represents an internal map  $f^* : \mathcal{O}(\mathcal{A}_{/\mathcal{X}}) \rightarrow \mathcal{O}(\mathcal{B}_{/\mathcal{X}})$  iff it is *extensional* in the sense that,

$$\text{for all } U, V \in \mathcal{O}(\mathcal{A}), \text{ we have } \llbracket U = V \rrbracket \leq \llbracket f^*U = f^*V \rrbracket. \quad (5)$$

Extensional maps correspond to commuting triangles  $\pi_{\mathcal{A}}^* = \pi_{\mathcal{B}}^* f^*$ .

**Lemma 3.** *For any extensional map*

$$f^* : \mathcal{O}(\mathcal{A}_{/\mathcal{X}}) \rightarrow \mathcal{O}(\mathcal{X}_{/\mathcal{X}}), \text{ we have } f^*(V) \wedge p \leq f^*(V \wedge \pi_{\mathcal{A}}^*(p)). \quad (6)$$

*Proof.* It follows from (3) that,  $p \leq \llbracket U = V \rrbracket$  iff  $U \wedge \pi_{\mathcal{A}}^*(p) = V \wedge \pi_{\mathcal{A}}^*(p)$ . Since  $p \leq \llbracket V = V \wedge \pi_{\mathcal{A}}^*(p) \rrbracket$ , we have  $p \leq \llbracket f^*(V) = f^*(V \wedge \pi_{\mathcal{A}}^*(p)) \rrbracket$ , and thus,  $f^*(V) \wedge p \leq f^*(V \wedge \pi_{\mathcal{A}}^*(p))$ . This is the semantic counterpart to Lemma 2.  $\square$

**Proposition 1.** [12] *The locale morphism  $\pi_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{X}$  is open (and surjective) iff the  $\mathcal{O}(\mathcal{X})$ -valued poset  $\mathcal{O}(\mathcal{A}_{/\mathcal{X}})$  it represents is internally open (and surjective).*

**Proposition 2.** *An element  $U \in \mathcal{O}(\mathcal{X})$  is said to be positive ( $\text{Pos}(U)$ ) iff every cover of  $U$  is inhabited. A locale,  $\mathcal{X}$ , is surjective iff  $\text{Pos}(\top)$ , and open iff  $\{U \mid \text{Pos}(U)\}$  is a basis for  $\mathcal{O}(\mathcal{X})$ .<sup>2</sup>*

**Formal Spaces.** are locales presented as spaces of models for some, possibly infinitary, geometric propositional theory. If  $x : \mathbb{1} \rightarrow \mathcal{X}$  is a point of  $\mathcal{X}$ , then for each  $\mathcal{U} \in \mathcal{O}(\mathcal{X})$ , we write  $x \in \mathcal{U}$  to mean that  $* \in x^*(\mathcal{U})$ , so  $x^*(\mathcal{U}) = \llbracket x \in \mathcal{U} \rrbracket$ . We can use the same notation,  $\llbracket \alpha \in \mathcal{U} \rrbracket = \alpha^*(\mathcal{U})$ , for a *generalised point*,  $\alpha$ , which is just a morphism  $\alpha : \mathcal{A} \rightarrow \mathcal{X}$ .

Consider a language  $\mathcal{L}$  with a set of basic propositions  $p \in \mathbb{P} \subseteq \mathcal{O}(\mathcal{X})$ . An  $\mathcal{O}(\mathcal{A})$ -valued model for  $\mathcal{L}$  is given by a morphism  $\alpha : \mathcal{A} \rightarrow \mathcal{X}$ . We give each basic proposition  $p \in \mathbb{P}$  the truth value  $\llbracket p \rrbracket_{\alpha} = \alpha^*(p) = \llbracket \alpha \in p \rrbracket$ . We say a *sequent*,  $p \vdash C$ , where  $p \in \mathbb{P}$  and  $C \subseteq \mathbb{P}$ , is *valid* for  $\alpha$  iff  $\llbracket p \rrbracket_{\alpha} \leq \bigvee \{\llbracket q \rrbracket_{\alpha} \mid q \in C\}$ .

**Definition 3.** [10] *A geometric presentation of a formal space  $(\mathbb{P}, \mathcal{A})$  consists of a structure  $\mathbb{P}$  of basic propositions and a collections  $\mathcal{A}$  of axioms:*

$\mathbb{P}$  is a preordered set with conditional finite meets: if a finite set has a lower bound then it has a greatest lower bound. In particular, if  $\mathbb{P}$  is inhabited, then it has a top element  $\top$ . We write  $p \downarrow$  for  $\{q \mid q \leq p\}$ . A *crible* of  $p$  is a set  $K \subseteq p \downarrow$ , such that  $\forall q \in K. q \downarrow \subseteq K$ . For  $K$  a crible of  $p$  and  $q \leq p$ , observe that,  $K \restriction q = K \cap q \downarrow$  is a crible of  $q$ .

$\mathcal{A}$  is a covering relation, that is, a set of sequents,  $p \vdash C$ , “ $C$  is a basic cover of  $p$ ”, where  $p \in \mathbb{P}$  and  $C \subseteq p \downarrow$ , which is stable in the sense that, if  $p \vdash C$ ,  $K$  is a crible of  $p$  with  $C \subseteq K$ , and  $q \leq p$ , then there is some basic cover  $q \vdash D$  of  $q$  such that  $D \subseteq K \restriction q$ .

<sup>2</sup> These appeared in [10] but are due to Joyal ([12] Chapter V;  Lemma C3.1.7).

A *crible*,  $K$  of  $\top$  is closed iff for all basic covers  $p \vdash C$ , if  $C \subseteq K$  then  $p \in K$ . The closed cribles are the formal opens  $\mathcal{O}(\mathbb{P}, \mathcal{A})$  of the formal space.

We say the formal space is separable if  $\mathbb{P}$  is countable and has decidable equality.

$\mathcal{O}(\mathbb{P}, \mathcal{A})$  is a frame. The corresponding locale is the formal space  $(\mathbb{P}, \mathcal{A})$  of models of the presentation. Geometrically a sequent,  $p \vdash C$ , is a cover; logically we read it as an entailment, where the right-hand side is an implicit disjunction. When presenting a formal space we often write a cover as a formal disjunction—this can be viewed as simply a suggestive notation for a set of basic propositions.

**Proposition 3.** *The formal space  $(\mathbb{P}, \mathcal{A})$  is open if for every cover  $p \vdash C \in \mathcal{A}$ ,  $C$  is inhabited. In this case, if  $\mathbb{P}$  is inhabited the space is surjective.*

**Definition 4.** *A  $\mathcal{O}(\mathcal{X})$ -valued model of  $(\mathbb{P}, \mathcal{A})$  is an assignment of a truth value  $\llbracket p \rrbracket \in \mathcal{O}(\mathcal{X})$  to each basic proposition such that:*

$$p \leq q \rightarrow \llbracket p \rrbracket \leq \llbracket q \rrbracket \quad \llbracket \top \rrbracket = \top \quad \llbracket p \rrbracket \wedge \llbracket q \rrbracket \leq \bigvee \{ \llbracket p \wedge q \rrbracket \} \quad (7)$$

$$\llbracket p \rrbracket \leq \bigvee \{ \llbracket q \rrbracket \mid q \in C \} \text{ for each axiom } p \vdash C \in \mathcal{A}. \quad (8)$$

Morphisms,  $\alpha^*: \mathcal{O}(\mathbb{P}, \mathcal{A}) \rightarrow \mathcal{O}(\mathcal{X})$ , from a locale  $\mathcal{X}$  to a formal space  $(\mathbb{P}, \mathcal{A})$  correspond to  $\mathcal{O}(\mathcal{X})$ -valued models. Since each  $p \in \mathbb{P}$  is a basic open of the formal space, we write  $\llbracket \alpha \in p \rrbracket$  for  $\alpha^*(p)$ .

*Examples.* For each example below, the basic opens are well-known from mathematical practice, and we adjust our notation accordingly — for example, to axiomatise a real number, we write  $p < r < q$  in place of  $r \in (p, q)$ .

For any set  $X$ , the *discrete formal space*,  $\mathcal{X}$ , is given by  $\mathbb{X} = X^\top$ , the poset obtained by adjoining a (new) top element,  $\top$ , with  $\forall x \in X. x < \top$ , together with a single axiom:

$$\mathcal{X} \quad \top \vdash \bigvee_{x \in X} \alpha = x \quad (9)$$

Each basic open is a singleton; we write  $\alpha = x$  for the basic proposition  $x$ . The corresponding frame is the power set,  $\mathcal{O}(\mathbb{X}, \top \vdash X) = \mathcal{P}(X)$ . A  $\mathcal{P}(\mathbb{1})$ -valued model, corresponds to a point of the formal space of models of  $(\mathbb{P}, \mathcal{A})$ . A  $\mathcal{P}(X)$ -valued model corresponds to a function: an  $X$ -indexed family of models.

If  $(\mathbb{P}, \mathcal{A})$  and  $(\mathbb{Q}, \mathcal{B})$  are formal spaces, their *product* is given by  $(\mathbb{P} \times \mathbb{Q}, \mathcal{A} + \mathcal{B})$  where  $\mathbb{P} \times \mathbb{Q}$  has the product (pointwise) preorder and  $\mathcal{A} + \mathcal{B}$  includes both  $\{(p, q) \vdash C \times \{q\} \mid p \vdash_{\mathcal{A}} C\}$  and  $\{(p, q) \vdash \{p\} \times C \mid q \vdash_{\mathcal{B}} C\}$ .

If  $\mathcal{X} = (\mathbb{P}, \mathcal{A})$  is a formal space, and  $X$  is a set then the formal product space is  $\mathcal{X}^X = (X \otimes \mathbb{P}, X \otimes \mathcal{A})$ . We introduce a formal  $\alpha : X \rightarrow \mathcal{X}$ . The basic proposition  $(x, p)$  should be read as  $x \in \alpha^*(p)$ .

Here,  $X \otimes \mathbb{P}$  consists of those finite subsets  $F \subseteq X \times \mathbb{P}$  satisfying the *compatibility condition*:  $(x, p) \in F \wedge (x, q) \in F \rightarrow (x, p \wedge q) \in F$ , ordered by,

$$F \leq G \text{ iff } \forall (x, p) \in G. p = \top \vee \exists q \leq p. (x, q) \in F. \quad (10)$$



We say  $(x, p)$  is *compatible with*  $F \in X \otimes \mathbb{P}$  iff for every  $q$  such that  $(x, q) \in F$  the meet  $p \wedge q$  is defined. In this case, we write

$$F \oplus (x, p) \text{ for } F \cup \{(x, p)\} \cup \{(x, p \wedge q) \mid (x, q) \in F\}.$$

$X \otimes \mathcal{A}$  includes a family of covers for each cover  $p \vdash C \in \mathcal{A}$ . For each  $(x, p)$  compatible with  $F$  we have a cover  $F \oplus \{(x, p)\} \vdash \{\{F \oplus \{q\} \mid q \in C\}\}$ . This constructive presentation of a product of locales is a (very) special case of Hyland's construction of exponents [21] Proposition 3: a discrete space is locally compact!

Formal *Baire Space*,  $\mathcal{B}$ , is the formal space of models of the theory of a function  $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ ; for brevity we call it simply the formal space of functions  $\alpha \in \mathbb{N}^{\mathbb{N}}$ . Similarly, formal *Cantor Space*,  $\mathcal{C}$ , is the formal space of functions  $\alpha \in \mathbf{2}^{\mathbb{N}}$ . The basic opens correspond to finite initial segments of an infinite sequence:  $a \prec \alpha$  is the proposition represented by  $a \in \mathbf{T} = \mathbb{X}^{<\mathbb{N}}$ , the tree of finite sequences, where  $\mathbb{X}$  is  $\mathbb{N}$  or  $\mathbf{2}$ , respectively. In any model,  $\top \vdash \langle \rangle \prec \alpha$ , since  $\langle \rangle = \top$ . We also require:

$$\mathcal{B} \quad a \prec \alpha \vdash \bigvee_{n \in \mathbb{N}} a \hat{\cdot} n \prec \alpha \quad \text{for each } a \in \mathbf{T} = \mathbb{N}^{<\mathbb{N}}, \quad (11)$$

$$\mathcal{C} \quad a \prec \alpha \vdash a \hat{\cdot} 0 \prec \alpha \vee a \hat{\cdot} 1 \prec \alpha \quad \text{for each } a \in \mathbf{T} = \mathbf{2}^{<\mathbb{N}}, \quad (12)$$

where,  $a \hat{\cdot} n$  is the extension of  $a$  by  $n$ . So, a model corresponds to an infinite path through the tree. We could, of course, construct these as exponents of discrete spaces.


The formal *Dedekind Reals*,  $\mathcal{R}$ , axiomatise an open cut in the rationals. Our basic propositions are proper, rational open intervals,  $(p, q)$  with  $p < q$ , where  $p, q \in \mathbb{Q}^* = \mathbb{Q} \cup \{-\infty, \infty\}$ . These intervals are ordered by inclusion. We write  $p < r < q$ , for  $r \in (p, q)$ . The covering axioms are:

$$\mathcal{R} \quad \begin{aligned} & p < r < q \vdash \bigvee \{p' < r < q' \mid p < p' < q' < q\} \\ & p < r < q \vdash p < r < q' \vee p' < r < q \text{ where, } p < p' < q' < q \end{aligned} \quad (13)$$

**Definition 5.** [17] A locale  $\mathcal{A}$  is  $T_1$  iff for every locale  $\mathcal{X}$ , the specialisation ordering on the set of  $\mathcal{X}$ -valued points  $[\mathcal{X}, \mathcal{A}]$  is trivial:  $x \leq y \rightarrow x = y$ , or, equivalently, if every localic topos,  $Sh(\mathcal{X})$  satisfies

$$\forall x, y \in Pt(\mathcal{A}). \forall U \in \mathcal{O}(\mathcal{A}). (x \in U \rightarrow y \in U) \rightarrow x = y.$$

**Lemma 4.** Each of  $\mathcal{X}, \mathcal{B}, \mathcal{C}, \mathcal{R}$  is  $T_1$ . if  $\mathcal{A}$  is  $T_1$  then so is  $\mathcal{A}^X$  for any set  $X$ .

*Proof.* For Baire space and Cantor space this is straightforward, since the values  $\llbracket \alpha(n) = m \rrbracket$ , where  $m, n \in \mathbb{N}$ , determine  $\alpha$ , which is single-valued.  Lemma 1.2.17 tells us that since  $\mathcal{R}$  is regular, it is  $T_1$  (there called  $T_U$ ), but a direct constructive proof “in the internal logic” is instructive.

Suppose  $s \leq r$ , are generalised points of  $\mathcal{R}$ : that is, for any  $U \in \mathcal{O}(\mathcal{R})$ , if  $s \in U$  then  $r \in U$  (it suffices to assume this for every proper rational interval

$\mathcal{U}$ ). For  $p < q < q' < p'$ , let  $\mathcal{P} = (p, p')$  and  $\mathcal{Q} = (q, q')$ . It suffices to show that, if  $r \in \mathcal{Q}$  then  $s \in \mathcal{P}$ , since such proper subintervals cover  $\mathcal{P}$ . Let  $\mathcal{W}$  be such that  $\mathcal{W} \wedge \mathcal{Q} = \perp$  and  $\mathcal{W} \vee \mathcal{P} = \top$  in  $\mathcal{O}(\mathcal{R})$ . (if such an  $\mathcal{W}$  exists we say  $\mathcal{Q} \triangleleft \mathcal{P}$ ; in this case, it can be chosen as the join of two basic intervals.) Certainly  $s \in \mathcal{P}$  or  $s \in \mathcal{W}$ , since  $\mathcal{W} \vee \mathcal{P} = \top$ . So  $s \in \mathcal{P}$  or  $r \in \mathcal{W}$ . Now suppose  $r \in \mathcal{Q}$ ; this is incompatible with  $r \in \mathcal{W}$ , since  $\mathcal{W} \wedge \mathcal{Q} = \perp$ , so we conclude that  $s \in \mathcal{P}$ .

Tracing the interpretation of this argument would give an algebraic proof: a sequence of inequalities starting from the assumption that  $\llbracket s \in \mathcal{U} \rrbracket \leq \llbracket r \in \mathcal{U} \rrbracket$  and showing for the chosen basic opens  $\mathcal{Q} \triangleleft \mathcal{P}$  that  $\llbracket r \in \mathcal{Q} \rrbracket \leq \llbracket s \in \mathcal{P} \rrbracket$ .  $\square$

**Lemma 5.** *If  $\mathcal{A} = (\mathbb{P}, \mathcal{A})$  is a formal space, and we lift the presentation to  $Sh(\mathcal{X})$  then the corresponding internal locale — the interpretation of the (lifting of the) presentation in  $Sh(\mathcal{X})$  — is represented by the projection  $\mathcal{X} \times \mathcal{A} \rightarrow \mathcal{X}$ , where  $\mathcal{X} \times \mathcal{A}$  is the product locale.*

**Proposition 4.** *If  $\mathcal{A} \rightarrow \mathcal{X}$  is a morphism of locales, then the corresponding geometric morphism  $Sh(\mathcal{A}) \rightarrow \mathbb{E} = Sh(\mathcal{X})$  is equivalent to the extension  $Sh(\mathcal{A}/\mathcal{X}) \rightarrow \mathbb{E} = Sh(\mathcal{X})$ .*

*Adjoint Retracts.* Now consider two formal spaces,  $\mathcal{P} = (\mathbb{P}, \mathcal{A})$  and  $\mathcal{Q} = (\mathbb{Q}, \mathcal{A})$ , where  $\mathbb{P} \subseteq \mathbb{Q}$  is a subset, with the inherited preorder, closed under conditional finite meets: if a finite subset of  $\mathbb{P}$  has a meet in  $\mathbb{Q}$  then its meet is in  $\mathbb{P}$ . These presentations have possibly different posets of basic propositions, but the same axioms, which must mention only propositions in  $\mathbb{P}$ . Clearly the map  $i^* : \mathcal{O}(\mathbb{Q}, \mathcal{A}) \rightarrow \mathcal{O}(\mathbb{P}, \mathcal{A})$ , given by  $V \mapsto V \cap \mathbb{P}$ , is a frame morphism, that also preserves  $\bigwedge$ . So it has both right ( $i_*$ ) and left ( $i_!$ ) adjoints:

$$\begin{array}{ccc} U \xrightarrow{i_*} \{q \mid \forall p \leq q. p \in U\} & & \mathcal{O}(\mathbb{P}, \mathcal{A}) \xrightleftharpoons[i_! = r^*]{i_* = r_*} \mathcal{O}(\mathbb{Q}, \mathcal{A}) \\ U \xrightarrow{i_!} \{q \mid \exists p \in U. q \leq p\} & & \end{array}$$

**Lemma 6.** [18] *In the situation just described,  $i_!$  preserves  $\bigwedge$ , so we have an adjoint retraction  $\mathcal{P} \xrightarrow{i_!} \mathcal{Q}$  of locales. For any  $T_1$  locale,  $\mathcal{X}$ , we have an equivalence  $\mathcal{L}[\mathcal{P}, \mathcal{X}] \cong \mathcal{L}[\mathcal{Q}, \mathcal{X}]$ .*

### 3 Reflections

A reflection principle in set theory asserts that some property of the class of all sets is *reflected* already in some set, and thus serves to extend the universe of discourse and reduce incompleteness. A proto-example might be the introduction of an infinite set by reflection on the closure of the class of all sets under the successor operation  $x \mapsto x \cup \{x\}$ . (See e.g. [22] for more elevated examples.)

Brouwer's introspection serves a similar philosophical purpose. It is natural to ask whether iterating Brouwer's second act leads to further insights. We say

that an extension is reflexive if truth in the iterated model is reflected to the model, as described in the Introduction.

### 3.1 Topological Models

Joyal first pointed out that topological models are best viewed as localic models that introduce a *generic point* of a formal space. From this perspective, Scott's topological model [1, 2] is an extension constructed by adding a generic point of  $\mathbb{N}^{\mathbb{N}}$ . From the classical perspective adopted in Scott's two papers there is no difference between the open sets of the space of points of  $\mathbb{N}^{\mathbb{N}}$ , equipped with the product topology, and the formal Baire space  $\mathcal{B}$  [10]. Here we start from a non-classical base. We take the formal space as the primary object of study.

Classically, the theories  $\mathcal{R}, \mathcal{B}, \mathcal{C}$  are complete — which means, in each case, that the formal space has enough points (to distinguish the formal opens), or equivalently that the topological opens and formal opens coincide. Constructively, this is not provable in HAH—completeness is equivalent (in HAH), for  $\mathcal{R}$ , to the Heine-Borel theorem ( $\mathbb{R}$  is locally compact), and for  $\mathcal{B}, \mathcal{C}$  to Brouwer's *Principle of Bar Induction*, and *Fan Theorem*, respectively [10].

**Theorem 1.** *The  $\mathcal{O}(\mathcal{X})$ -valued model includes sufficient points to distinguish the formal opens of  $\mathcal{X}$ , where  $\mathcal{X}$  may be  $\mathcal{B}, \mathcal{C}$ , or  $\mathcal{R}$ .*

*Proof.* Our proof is constructive, and does not presume a metatheory in which  $\mathcal{X}$  has enough points. Let  $\mathcal{O}(\mathcal{X}) = \mathcal{O}(\mathbb{P}, \mathcal{A})$ . In the  $\mathcal{O}(\mathcal{X})$ -valued model,  $\mathcal{X}$  is represented by a projection  $\pi : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ . Elements of  $\mathbb{P}$  play two rôles: in the first dimension (onto which we project),  $p \in \mathbb{P}$  is a basic truth value. In the second dimension  $q \in \mathbb{P}$  is a basic proposition of the internal presentation of  $\mathcal{X}$ . An internal formal open  $K \in \mathcal{O}(\mathcal{X})$  is represented by a formal open of  $\mathcal{X} \times \mathcal{X}$ , determined by the values  $\llbracket q \in K \rrbracket$ , for  $q \in \mathbb{P}$ , which, in turn, are determined by the sets  $\{p \in \mathbb{P} \mid p \leq \llbracket q \in K \rrbracket\}$ .

Internal points are functions  $\alpha : \mathcal{X} \rightarrow \mathcal{X}$ , or, equivalently, sections of the projection [16]. The identity function on  $\mathcal{X}$  (the diagonal section) gives a *generic point*,  $\gamma$ . By definition,  $\llbracket \gamma \in q \rrbracket = q \downarrow$ , for any  $q \in \mathbb{P}$ .

To show that  $\mathcal{X}$  has enough points it suffices to exhibit points  $\alpha_{p,q}$ , where if  $q = \top$  then  $p = \top$ , such that  $\llbracket \alpha_{p,q} \in \bar{q} \rrbracket = p \downarrow$ ; that is, frame morphisms  $\alpha_{a,b}^* : \mathcal{X} \rightarrow \mathcal{X}$  such that  $\alpha_{p,q}^*(q \downarrow) = p \downarrow$ . In the case of the reals, for example, there is a unique rational linear function that maps one rational open interval to another; so these functions suffice to distinguish formal opens. We leave  $\mathcal{B}$  and  $\mathcal{C}$  as an exercise for the reader.  $\square$

Whether the base from which we start is classical or constructive, the localic model using the formal opens produces an extension  $\pi : \mathbb{B}[\mathcal{B}] \rightarrow \mathbb{B}$  that includes a generic point, corresponding to the identity morphism  $\gamma : \mathcal{B} \rightarrow \mathcal{B}$ . Geometrically, *generic* means that the points  $\alpha$  of  $\mathcal{B}$  in any topos  $\mathbb{E} \rightarrow \mathbb{B}$  correspond to geometric morphisms,  $\mathbb{E} \xrightarrow{a} \mathbb{B}[\mathcal{B}]$ , with  $\alpha = a^*(\gamma)$ , making the triangle commute.

$$\begin{array}{ccc}
\mathbb{E} & \xrightarrow{a} & \mathbb{B}[\mathcal{B}] \\
& \searrow & \swarrow \kappa \\
& \mathbb{B} &
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{B}[\mathcal{B}][\mathcal{B}] & \xrightarrow{d} & \mathbb{B}[\mathcal{B}] \\
\downarrow b & & \downarrow \pi \\
\mathbb{B}[\mathcal{B}] & \xrightarrow{\pi} & \mathbb{B}
\end{array}$$

Iterating this construction gives us a topos  $\mathbb{B}[\mathcal{B}][\mathcal{B}] \xrightarrow{b} \mathbb{B}[\mathcal{B}]$ . Like any topos over  $\mathbb{B}[\mathcal{B}]$  it includes a point  $\beta = b^* \gamma \in \mathcal{B}$ ; it also includes another point  $\delta \in \mathcal{B}$  which is generic for points of  $\mathcal{B}$  in toposes over  $\mathbb{B}[\mathcal{B}]$ . Since any topos over  $\mathbb{B}[\mathcal{B}]$  is also a topos over  $\mathbb{B}$ , we see that  $\delta$  corresponds to a morphism  $\mathbb{B}[\mathcal{B}][\mathcal{B}] \xrightarrow{d} \mathbb{B}[\mathcal{B}]$  making the square commute. Furthermore, the square is a pullback, by the universal property of our second extension. Logically,  $\mathbb{B}[\mathcal{B}][\mathcal{B}] \rightarrow \mathbb{B}$  classifies pairs of models of the formal space  $\mathcal{B}$ . Geometrically, it is given by the formal space  $\mathcal{B} \times \mathcal{B}$  whose points are pairs of points of  $\mathcal{B}$ .

Classically, it is well-known that  $\mathcal{B} \times \mathcal{B} \cong \mathcal{B}$ . The classical proof exhibits a homeomorphism  $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \cong \mathbb{N}^{\mathbb{N}}$ , for example, a **zip** function that interleaves two sequences, whose inverse takes  $\alpha$  to the pair  $(\text{even}(\alpha), \text{odd}(\alpha))$ . So the double extension is equivalent to the single extension, and has the same logic. Working constructively, the same is true, but we must work directly with the formal opens. The map  $a \downarrow \mapsto (\text{even}(a), \text{odd}(a)) \downarrow$  gives a homeomorphism of formal spaces.

Entirely analogous remarks hold for formal Cantor space,  $\mathcal{C}$ , *mutatis mutandis*, with  $\mathbf{T} = \mathbf{2}^{\mathbb{N}}$ . So we have full reflection for [5] (2.1) Open Data.

**Proposition 5.** *If  $\mathbb{B}$  is an elementary topos,  $\mathcal{B}$  the formal Baire space, and  $\mathcal{C}$  the formal Cantor space in  $\mathbb{B}$ , then  $\mathbb{B}[\mathcal{B}][\mathcal{B}] \equiv \mathbb{B}[\mathcal{B}]$  and  $\mathbb{B}[\mathcal{C}][\mathcal{C}] \equiv \mathbb{B}[\mathcal{C}]$  as toposes over  $\mathbb{B}$ . So,  $\mathbb{B}[\mathcal{B}] \models \phi$  iff  $\mathbb{B}[\mathcal{B}] \models \ulcorner \mathbb{B}[\mathcal{B}] \models \phi \urcorner$ , for any formula  $\phi$  of HAH.*

We have no such straightforward reflection theorem for  $\mathcal{R}$ , since  $\mathcal{R} \neq \mathcal{R} \times \mathcal{R}$ .

### 3.2 Extensions over Sites

**Definition 6.** [12] A site  $(\mathbb{C}, \mathcal{J})$  is a category  $\mathbb{C}$  equipped with a covering system,  $\mathcal{J}$ . That is, a collection,  $\mathcal{J}(A)$ , of covers  $R = \{A_i \rightarrow A\}_{i \in I}$ , for each object  $A$  in  $\mathbb{C}$ , such that: if  $\alpha : A' \rightarrow A$  and  $R \in \mathcal{J}(A)$ , then, for some  $R' \in \mathcal{J}(A')$  we have  $R' \subseteq \{\beta : A'' \rightarrow A' \mid \beta\alpha \in R\}$ .

*The Fundamental Fibration.* [15] Let  $\mathbf{T} = (\mathbb{C}, \mathcal{J})$  be a site, where  $\mathbb{C}$  has pullbacks. The topos of sheaves can be viewed as an extension  $\mathbb{B}[\mathbb{C}, \mathcal{J}] = \text{Sh}(\mathbb{C}, \mathcal{J})$ : the Yoneda embedding provides a universal model of  $(\mathbb{C}, \mathcal{J})$  in a topos over our base,  $\mathbb{B}$ . Grothendieck showed how the Yoneda embedding  $Y : \mathbb{C} \rightarrow \mathbb{B}[\mathbb{C}, \mathcal{J}]$  can be viewed as an internal site. We view the codomain projection  $\mathbb{C}^2 \rightarrow \mathbb{C}$ , from the category of arrows  $\mathbb{C}^2$  to  $\mathbb{C}$  as an internal category,  $\mathbb{C}_{/\mathbb{C}}^2$ , whose fibre over a representable  $X$  is the slice category  $\mathbb{C}/X$ . For  $\alpha : Y \rightarrow X$ , the restriction map  $\alpha^* : \mathbb{C}/X \rightarrow \mathbb{C}/Y$  is given by pullbacks along  $\alpha$ .

We have described the internal category  $\mathbb{C}_{/\mathbb{C}}^2$  corresponding to the fibration  $\partial_1$ . The topology  $\mathcal{J}$  also lifts to an internal topology  $\mathcal{J}_{/\mathbb{C}}$  on  $\mathbb{C}_{/\mathbb{C}}^2$ : if  $R = \{u_i : A_i \rightarrow A\}_{i \in I}$  is a covering family in  $\mathbb{C}$ , and  $\beta : A \rightarrow X \in \mathbb{C}/X$ , then  $R\beta = \{u_i : u_i\beta \rightarrow \beta\}_{i \in I}$  covers in  $\mathbb{C}/X$ .

To give an external representation of this extension, let  $\mathcal{J}^2$  be the topology on  $\mathbb{C}^2$  with covering families as follows: If  $R = \{u_i : A_i \rightarrow A\}_{i \in I}$  is a covering family in  $\mathbb{C}$ , and  $\beta : A \rightarrow X \in \mathbb{C}/X$ , then  $R\beta = \{u_i : u_i\beta \rightarrow \beta\}_{i \in I}$  covers  $\beta : A \rightarrow X$ . If, furthermore,  $\delta : X \rightarrow A$  then the pullbacks  $\delta^*u_i : A_i \times_A X \rightarrow X$  below cover  $\delta : X \rightarrow A$ :

$$\begin{array}{ccc}
 A_i & \xrightarrow{u_i} & A \\
 & \searrow & \swarrow \beta \\
 & X &
 \end{array}
 \qquad
 \begin{array}{ccc}
 A_i \times_A X & \xrightarrow{\delta^*u_i} & X \\
 \downarrow & \lrcorner & \downarrow \delta \\
 A_i & \xrightarrow{u_i} & A
 \end{array}
 .$$

The codomain morphism  $\partial_1 : \mathbb{C}^2 \rightarrow \mathbb{C}$  gives a geometric morphism,

$$\partial_1 : \text{Sh}(\mathbb{C}^2, \mathcal{J}^2) \rightarrow \text{Sh}(\mathbb{C}, \mathcal{J}),$$

whose inverse image is composition with  $\partial_1$  followed by sheafification; and whose direct image is given by composition with the inclusion  $\Delta : \mathbb{C} \rightarrow \mathbb{C}^2$ , which takes each object to its identity morphism. The inverse image just constructs internal constant sheaves, and the direct image takes global sections. Since  $\Delta$  preserves covers and  $\partial_1 \dashv \Delta$ , this is a case of Theorem 4 of M&M §VII.10.

### Proposition 6

*The geometric morphism  $\partial_1 : \text{Sh}(\mathbb{C}^2, \mathcal{J}^2) \rightarrow \mathbb{E} = \text{Sh}(\mathbb{C}, \mathcal{J})$  is equivalent to the extension  $\text{Sh}(\mathbb{C}_{/\mathbb{C}}^2, \mathcal{J}^*) \rightarrow \mathbb{E} = \text{Sh}(\mathbb{C}, \mathcal{J})$ .*

We now investigate the logical properties of the iterated extension. Just as in the localic case, it suffices to find some functor comparing  $\widetilde{\mathbb{C}^2}$  and  $\widetilde{\mathbb{C}}$ . We have three functors  $\partial_1 \dashv \Delta \dashv \partial_0$ . These all preserve covers, so  $\partial_1$  and  $\Delta$  have the cover lifting property (M&M §VII.10 Lemma 3).

**Lemma 7.**  *$\partial_0$  also has the cover lifting property.*

*Proof.* If  $\mathbf{R} = \{\mathbf{X}_i \rightarrow \mathbf{X}\}_{i \in I}$  is such that both  $\partial_0(\mathbf{R}) = \{\partial_0(\mathbf{X}_i) \rightarrow \partial_0(\mathbf{X})\}_{i \in I}$  and  $\partial_1(\mathbf{R})$ , defined similarly, are covers in  $\mathbb{C}$ , then  $\mathbf{R}$  is a cover in  $\mathbb{C}^2$ .

Suppose  $\mathbf{X} = \pi : X \rightarrow Y$  and  $R = \{u_i : X_i \rightarrow X\}_{i \in I}$  is a cover in  $\mathbb{C}$  then the morphisms,  $(u_i, \mathbb{1}_Y)$ , from the objects  $R/Y = \{u_i\pi : X_i \rightarrow Y\}_{i \in I}$  in  $\mathbb{C}^2$ , to  $\mathbf{X}$ , form a cover in  $\mathcal{J}^2$ .

We are now in the situation where each of the adjoint functors  $\partial_1 \dashv \Delta \dashv \partial_0$  preserves covers and has the cover lifting property. Therefore (M&M §VII.10), we

have three geometric morphisms whose inverse images are given by composition (e.g.  $\partial_0^*(A) = \partial_0 A$ ), followed by sheafification. We write  $\widetilde{\mathbb{C}}$  for  $\text{Sh}(\mathbb{C})$ .

$$\widetilde{\mathbb{C}}^2 \begin{array}{c} \xrightarrow{\partial_0} \\ \xleftarrow{\partial_1} \end{array} \widetilde{\mathbb{C}} \quad \widetilde{\mathbb{C}}^2 \xleftarrow{\partial_1^*} \widetilde{\mathbb{C}} \xleftarrow[\partial_{1*} = \partial_{0!}]{\Delta^*} \widetilde{\mathbb{C}}^2 \xleftarrow[\Delta_*]{\partial_0^*} \widetilde{\mathbb{C}} \quad (14)$$

Both  $\Delta^*$  and  $\partial_0^*$  preserve sheaves, so, for them, sheafification is unnecessary. Since  $\Delta\partial_0 = \mathbb{1}_{\mathbb{C}}$  we have an adjoint retraction  $\widetilde{\mathbb{C}} \xrightleftharpoons[\partial_0]{\Delta} \widetilde{\mathbb{C}}^2$ , and  $\partial_0$  is a surjection. We have  $\partial_{0!} = \Delta^* \dashv \partial_0^*$  and, since these functors preserve sheaves,

$$\Delta^*(B \times_{\widetilde{\mathbb{C}}^2} \partial_0^*(A)) = (\Delta B \times_{\widetilde{\mathbb{C}}} \Delta\partial_0 A) = \Delta B \times_{\widetilde{\mathbb{C}}} A = \Delta^*(B) \times_{\widetilde{\mathbb{C}}} A; \quad (15)$$

the Frobenius condition holds. This means that  $\partial_0$  is locally connected, hence open; it preserves exponentials and first-order logic. In fact, we are in the situation described by Moerdijk and Reyes [23] Theorem 2.2:  $\partial_0$  is a left-exact functor which preserves covers and has the covering lifting property;  $\Delta \dashv \partial_0$  is a left-adjoint “right inverse”,  $\Delta\partial_0 = \mathbb{1}_{\mathbb{C}}$ . This gives a principle of *predicative reflection*.

**Proposition 7.** [23]  $\partial_0^*$  preserves and reflects first-order logic, preserves exponentials, and preserves the sheaf of points of any  $T_1$  formal space.

So,  $\text{Sh}(\mathbb{C}, \mathcal{J}) \models \phi$  iff  $\text{Sh}(\mathbb{C}, \mathcal{J}) \models \ulcorner \text{Sh}(\mathbb{C}_{/\mathcal{X}}^2, \mathcal{J}^*) \models \phi \urcorner$ , for  $\phi$  a formula in a language for predicative analysis — a language with finite types over  $\mathbb{N}, \mathcal{R}$ , possibly with constants for relations and functions in  $\text{Sh}(\mathbb{C}, \mathcal{J})$ .

We might hope for an impredicative reflection, but this seems unlikely for extensions over sites. Extensions that preserve powersets are quite special.

**Lemma 8.** Let  $\mathbb{C}, \mathcal{J}$  be a site. If  $\Omega(B)$ , the frame of closed criples of some  $B \in \mathbb{C}_0$ , is isomorphic to a powerset  $\mathcal{P}(X)$ , then  $X$  is a singleton, and every inhabited sieve contains  $\mathbb{1}_B$ .<sup>3</sup>

*Proof.* Let  $\phi$  be the composite  $X \rightarrow \mathcal{P}(X) \rightarrow \Omega(B)$  be the composite of the singleton map with the isomorphism. Then, since  $X = \bigcup \{\{x\} \mid x \in X\}$ , we have  $\mathbb{1}_B \in \bigcup \{\phi(x) \mid x \in X\}$ . Thus, for some  $x \in X$  we have  $\mathbb{1}_B \in \phi(x)$ , so,  $\phi(x) = \top$ . Furthermore, given such an  $x$ , for all  $y \in X$  we have  $\phi(y) \leq \phi(x)$ , whence  $\{y\} \subseteq \{x\}$ ; ergo,  $y = x$ .  $\square$

**Theorem 2.** If  $\mathbb{E}^{\text{C}^{\text{op}}}$  is an atomic topos then  $\mathbb{C}$  is a groupoid in  $\mathbb{E}$ .<sup>4</sup>

*Proof.* This is a direct consequence of the lemma since Barr and Diaconescu show (op. cit.) that for each object  $A$  in an atomic topos  $\Gamma(\Omega^A)$  is isomorphic to some  $\mathcal{P}(X)$ .  $\square$

<sup>3</sup> A classical proof of this fact appears in [24] (§7 Example 2). Here we give a constructive proof that can be interpreted in any elementary topos.

<sup>4</sup> This is implicit in [12] VII.4. Barr and Diaconescu use their classical version of the lemma to prove this for a Boolean base topos,  $\mathbb{E}$ . (Corollary C3.5.2).

### 3.3 Continuous Truth

**Definition 7.** A topological site is a category of open locales and continuous maps, including enough open inclusions (a basis for each locale), closed under finite limits, and equipped with the open cover topology.

This definition differs from those of [18, 23]; Moerdijk and Reyes use topological spaces to construct their topological sites; we use locales. They, therefore, have to appeal to principles such as Bar Induction, or Fan Theorem, in the metatheory in order to show they hold in the topos of sheaves.

In [6] §4, we claimed, with proofs formalisable in HAH, that general principles of continuity, local choice and local compactness hold for these models. The proof of the key result, Proposition 4.1, presumes that a projection  $\pi : \mathcal{W} \times \mathcal{U} \longrightarrow \mathcal{U}$  is a cover for the open cover topology. This is true if  $\mathcal{W}$  is an open surjective locale, but obviously not in general—consider the empty space. The remedy is to require that the site  $(\mathbb{C}, \mathcal{J})$  introduced in the opening sentence of §4 should be a topological site, whose objects are *open* locales. The results claimed in §4 are then valid if we take any elementary topos with natural number object as a base. We restate and prove Proposition 4.1.

**Theorem 3.** Let  $(\mathbb{C}, \mathcal{J})$  be a topological site. For any  $\mathcal{X} \in \mathbb{C}$ , the internal locale,  $\mathfrak{X}$  represented by,  $\mathcal{O}(\mathfrak{X})(\mathcal{U}) = \mathcal{O}(\mathcal{U} \times \mathcal{X})$ , has enough points.

*Proof.* We must show that, if  $\mathcal{U} \Vdash \mathcal{K}$  covers  $\text{Pt}(\mathcal{W})$  and  $\mathcal{U} \Vdash \mathcal{K}$  is closed, then  $\mathcal{U} \Vdash \mathcal{W} \in \mathcal{K}$ , for  $\mathcal{U} \in \mathbb{C}$ , and  $\mathcal{W} \in \mathcal{O}(\mathfrak{X})$ . We assume the hypotheses, and let

$$\mathbb{K} = \{\mathcal{U}_i \times \mathcal{W}_i \mid \mathcal{U}_i \Vdash \mathcal{W}_i \in \mathcal{K} \upharpoonright \mathcal{U}_i\} \quad (16)$$

Clearly,  $\mathbb{K}$  is a closed crible of  $\mathcal{O}(\mathcal{U}) \times \mathcal{O}(\mathcal{X})$ , that is, an open in  $\mathcal{O}(\mathcal{U} \times \mathcal{X})$ . We will show that  $\mathbb{K}$  covers  $\mathcal{U} \times \mathcal{W}$ , so  $\mathcal{U} \times \mathcal{W} \in \mathbb{K}$ , which means that  $\mathcal{U} \Vdash \mathcal{W} \in \mathcal{K}$ . We pull back along the projection  $\pi_2 : \mathcal{W} \times \mathcal{U} \longrightarrow \mathcal{U}$ . This introduces a generic point of  $\mathcal{W}$  given by  $\pi_1 : \mathcal{W} \times \mathcal{U} \longrightarrow \mathcal{W}$ . We have,  $\mathcal{W} \times \mathcal{U} \Vdash \mathcal{K} \upharpoonright \pi_2$  covers  $\text{Pt}(\mathcal{W})$ , by persistence. In particular, the generic point is covered:

$$\mathcal{W} \times \mathcal{U} \Vdash \mathcal{K} \upharpoonright \pi_2 \text{ covers } \pi_1 \quad \text{that is, } \mathcal{W} \times \mathcal{U} \Vdash \exists V \in \mathcal{K} \upharpoonright \pi_2. \pi_1 \in V.$$

$V$  ranges over basic opens of  $\mathcal{X}$ . By unpacking the forcing definition, we see that

$$\mathbb{K}^* = \{\mathcal{W}_i \times \mathcal{U}_i \mid \text{for some } V \leq \mathcal{W}, \mathcal{W}_i \times \mathcal{U}_i \Vdash V \in \mathcal{K} \upharpoonright \pi_2 \wedge \pi_1 \in V\} \quad (17)$$

covers  $\mathcal{W} \times \mathcal{U}$ . We now show that every basic open  $W \times U \in \mathbb{K}^*$  is in  $\mathbb{K}$ . Given  $U, V, W$  such that  $W \times U \Vdash V \in \mathcal{K} \upharpoonright \pi_2 \wedge \pi_1 \in V$  we must show  $U \Vdash W \in \mathcal{K}$ . First,  $W \times U \Vdash \pi_1 \in V$  iff  $W \leq V$ , so, by monotonicity,  $W \times U \Vdash W \in \mathcal{K} \upharpoonright \pi_2$ . Second,  $W$  is open surjective, so  $\pi_2$  is an open surjection; viewed as a basic open in  $\mathcal{O}(\mathfrak{X})$ ,  $W$  is constant, so  $U \Vdash W \in \mathcal{K}$ .  $\square$

We observe that this proof does not require that every object of  $\mathbb{C}$  be surjective, indeed a subcategory of  $\mathcal{L}$  with only open surjective objects will seldom be closed under limits. However, if  $W$  is a positive open of any open space, like the  $W$  in

the final steps of the proof just given, then  $W$ , as a subspace in its own right, is open surjective. Any open space is surjective in so far as its positive basis is inhabited. Nor do we require a full subcategory of  $\mathcal{L}$ , so the result should apply, for example, to a suitable localic version of the Euclidean topos, of sheaves over the category of closed subspaces of  $\mathbb{R}^n$  with  $C^\infty$  functions, defined by Moerdijk and Reyes [23], and other smooth topoi.

**Proposition 8.** *If  $(\mathbb{C}, \mathcal{J})$  is a topological site and  $\mathcal{X} \in \mathbb{C}$ , then the inclusion functor  $i : \mathcal{O}(\mathcal{X}) \rightarrow \mathbb{C}/\mathcal{X}$  has a right adjoint  $\pi$  which induces an adjoint retract pair of geometric morphisms  $i : Sh(\mathcal{X}) \xrightleftharpoons[\pi]{i} Sh(\mathbb{C}/\mathcal{X})$ . So,  $\pi^*$  preserves and reflects first-order logic, preserves exponentials, and preserves the sheaf of points of any  $T_1$  formal space.*

*Proof.* The right adjoint  $\pi$  is given by  $f \mapsto \bigvee \{U \mid U \text{ factors through } f\}$ , which satisfies the conditions of [23] Theorem 2.2. So Proposition 7 applies. (The spatial counterpart of this proposition appears in M&M (§VII.10 Theorem 5 ff.).

To show a predicative reflection principle, we must choose a suitable topological site  $(\mathbb{C}, \mathcal{J})$  with extension  $\mathbb{E} = Sh_{\mathbb{B}}(\mathbb{C}, \mathcal{J})$ ; then provide a representation of the iterated extension  $Sh_{\mathbb{B}}(\mathbb{C}, \mathcal{J})$  as  $Sh_{\mathbb{B}}(\mathbb{C}^\dagger, \mathcal{J}^\dagger)$ , together with a left-exact functor  $\mathbb{C}^\dagger \rightarrow \mathbb{C}$  which preserves covers, and has the covering lifting property and a left-adjoint “right inverse”.

*Conjecture 1.* Let  $(\mathbb{C}, \mathcal{J})$  be the topological site of open subspaces of separable open locales with open maps, with the open cover topology, then  $(\mathbb{C}^2, \mathcal{J}^2)$  provides a representation of the corresponding internal site.

This general setting should provide reflection principles for several of the extensions introduced in [5]: (2.2) Independent Open Data, (2.3) Lawless Data, and (2.4) Spread Data. One key point is that the category of open spaces with open maps is closed under finite limits.

To extend such an account to more general examples, such as (2.5) Continuous Data, will require further analysis of the constructive theory of the category of open locales and continuous maps.

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